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A mathematical model of fracture phenomena on a spring-block system

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Abstract

We propose a crack propagation model on a spring-block system using an idea of phase field model for the damage of springs. We consider a discrete model of elastic body using a scalar or tensor-valued spring-block system, and study its properties in detail. Our fracture model is constructed on the spring-block system. It is described in a mathematically clear way and the unique existence and regularity of a solution are proved.

1 Introduction

For crack propagation and fracture phenomena, a number of engineering-oriented simulation algorithms, such as extended finite element method (X-FEM) [2], discrete element method (DEM) [3, 7], particle discretization scheme (PDS-FEM) [4, 5] etc., are widely used in engineering computing. On the other hand, from a viewpoint of mathematical analysis, it is difficult to prove some mathematical properties of the engineering-oriented models such as unique existence and energy estimates, since they are often not described in sufficiently mathematical ways. In this research, we construct a mathematical framework for a phase field model of material damage on a spring-block system. The obtained model is described in a mathematically clear way and admits some mathematical analysis.

The outline of this paper is as follows. In Section 2, we construct scalar and tensor-valued spring-block systems, which corresponds to anti-plane displacement and linear elasticity problems, respectively. Their mathematical properties such as solvability of a boundary value problem on the spring-block system are shown. In Section 3, we propose a mathematical model of fracture dynamics or crack propagation on the spring-block system by introducing a damage variable in Problem 3.3 and 3.4. We represent the fracture by giving damage to the spring constant and cutting the spring according to the damage. In Theorems 3.5, 3.6 and 3.7, we prove unique existence and regularity of a local solution and existence of a global solution.

2 Spring-block system

2.1 Block division

Let $n \in \mathbb{N}$ and let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary Γ. The outer unit normal vector on Γ is denoted by $\nu \in \mathbb{R}^n$. We denote the inner product and the norm of $L^2(\Omega)$ as

$$(u,v)_0 := \int_\Omega u(x)v(x)\,dx, \quad \|u\|_0 := \sqrt{(u,u)_0},$$

where u, v are real valued functions in $L^2(\Omega)$.

We divide Ω into N subblocks $\mathcal{D} = \{D_i\}_{i=1}^N$. We suppose that each block D_i is a nonempty connected open set in $\mathbb{R}^n$ and the conditions:

$$\overline{\Omega} = \bigcup_{i=1}^N \overline{D_i}, \qquad D_i \cap D_j = \emptyset \ \ (i \neq j).$$

If $n \geq 2$, we additionally suppose that D_i has a Lipschitz boundary, and denote the outer unit normal vector on ∂D_i by $\nu^i \in \mathbb{R}^n$. The n-dimensional volume of D_i is denoted by $|D_i|$. In this paper, for simplicity, we call $\mathcal{D} = \{D_i\}_{i=1}^N$ a block division of Ω and assume the above conditions.

We introduce the following notation for adjacent blocks in a block division $\mathcal{D}$.

$$\begin{aligned}
&D_{ij} := \overline{D_i} \cap \overline{D_j} \quad (i,\ j = 1,\dots,N,\ i \neq j),\\
&d_{ij} := \mathcal{H}^{n-1}(D_{ij}), \quad (i,\ j = 1,\dots,N,\ i \neq j),\\
&\Lambda_i := \{j\ ;\ d_{ij} > 0\} \quad (i = 1,\cdots,N),\\
&\Lambda := \{(i,j);\ 1 \leq i < j \leq N,\ d_{ij} > 0\},\\
&\Sigma := \bigcup_{(i,j)\in\Lambda} D_{ij},
\end{aligned} \tag{2.1}$$

where $\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure. In particular, for $(i,j) \in \Lambda$, the blocks D_i and D_j are adjacent and d_{ij} becomes

$$d_{ij} = \begin{cases} 1 & (n=1) \\ \text{length of } D_{ij} & (n=2) \\ \text{area of } D_{ij} & (n=3). \end{cases}$$

We define function spaces of piecewise constant on D_i and D_{ij} as follows.

$$\chi_i(x) := \begin{cases} 1 & (x \in D_i) \\ 0 & (x \in \Omega \setminus D_i) \end{cases} \qquad (i = 1,\dots,N)$$

$$\chi_{ij}(x) := \begin{cases} 1 & (x \in D_{ij}) \\ 0 & (x \in \Sigma \setminus D_{ij}) \end{cases} \qquad ((i,j) \in \Lambda)$$

$$V(\mathcal{D}) := \left\{ v \in L^\infty(\Omega);\ v = \sum_{i=1}^N v_i\chi_i,\ v_i \in \mathbb{R} \right\}$$

$$W(\mathcal{D}) := \left\{ \zeta \in L^\infty(\Sigma);\ \zeta = \sum_{(i,j)\in\Lambda} \zeta_{ij}\chi_{ij},\ \zeta_{ij} \in \mathbb{R} \right\}$$

In the following sections, we consider scalar or vector valued displacement field which belongs to $V(\mathcal{D})$, and virtual springs between adjacent blocks with a damage variable $z \in W(\mathcal{D})$.

In most of boundary value problems of linear elasticity, we have to set a Dirichlet boundary condition in a part of the boundary. Corresponding to the Dirichlet boundary condition, we suppose that

$$J = (J_0, J_1), \quad J_0 \cup J_1 = \{1, \ldots, N\}, \quad J_0 \cap J_1 = \emptyset, \quad J_0 \neq \emptyset, \quad J_1 \neq \emptyset,$$

and suppose that the balance of forces is considered at D_i for $i \in J_0$ and the displacement of D_i for $i \in J_1$ is a priori given. The displacement space $V(\mathcal{D})$ is a direct sum of the following subspaces:

$$V_l(\mathcal{D}) := \left\{ v \in V(\mathcal{D});\ v = \sum_{i \in J_l} v_i \chi_i,\ v_i \in \mathbb{R} \right\} \qquad (l = 0, 1).$$

2.2 Scalar spring constant model

For a block division $\mathcal{D}$ of Ω, a scalar valued spring-block system is constructed as follows. We consider $u = \sum_{i=1}^{N} u_i \chi_i \in V(\mathcal{D})$ and call $u_i \in \mathbb{R}$ a displacement of the block D_i.

In the case $n = 1$ or 2, our spring-block system has a physical interpretation as follows. In the space $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with a coordinate $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, in equilibrium, the n-dimensional object Ω is located on the hyperplane of $y = 0$, namely on the line $(n = 1)$ or on the plane $(n = 2)$. Under some body and boundary forces, we assume that the divided block D_i moves only into y-direction of the displacement $u_i \in \mathbb{R}$.

For a fixed i, the block D_j is adjacent to D_i if $j \in \Lambda_i$. We consider a virtual spring between D_i and D_j, and suppose that it has a spring constant $\kappa_{ij} > 0$, and suppose that the force acting on D_i from D_j is given as $\kappa_{ij}(u_j - u_i) \in \mathbb{R}$. This represents a sort of the Hook's law. From the action-reaction law, κ_{ij} should satisfy the condition:

$$\kappa_{ij} = \kappa_{ji} \geq 0 \qquad ((i, j) \in \Lambda).$$

We define $\kappa := \sum_{(i,j) \in \Lambda} \kappa_{ij} \chi_{ij} \in W(\mathcal{D})$. In this paper, under the above conditions, we call $(\mathcal{D}, \kappa)$ *a scalar spring-block system*, and call $(\mathcal{D}, \kappa, J)$ *a scalar spring-block system with Dirichlet boundary*.

We consider the following problem.

Problem 2.1. *Let $(\mathcal{D}, \kappa, J)$ be a scalar spring-block system with Dirichlet boundary in $\mathbb{R}^n$. For a given body force $f = \sum f_i \chi_i \in V_0(\mathcal{D})$ with $F_i := f_i |D_i|$ and a given displacement $g = \sum g_i \chi_i \in V_1(\mathcal{D})$, find a displacement $u = \sum u_i \chi_i \in V(\mathcal{D})$ such that*

$$\begin{cases} \displaystyle\sum_{j \in \Lambda_i} \kappa_{ij}(u_j - u_i) + F_i = 0 & (i \in J_0), \\ u_i = g_i & (i \in J_1). \end{cases} \tag{2.2}$$

The first equation of (2.2) represents the balance of force acting on the block D_i $(i \in J_0)$, and the second one represents the essential boundary condition of u_i for $i \in J_1$.

We introduce the following symmetric bilinear form and seminorm:

$$(u,v)_\kappa := \sum_{(i,j)\in\Lambda} \kappa_{ij}(u_j - u_i)(v_j - v_i) \quad (u,\ v \in V(\mathcal{D})), \tag{2.3}$$
$$|v|_\kappa := \sqrt{(v,v)_\kappa} \quad (v \in V(\mathcal{D})).$$

For Problem 2.1, we consider the following elastic energy of the springs with the outer force and an affine space for the Dirichlet boundary condition:

$$E(v) := \frac{1}{2}|v|_\kappa^2 - (f,v)_0 \qquad (v \in V(\mathcal{D})),$$
$$V(\mathcal{D},g) := \{v \in V(\mathcal{D});\ v - g \in V_0(\mathcal{D})\} \qquad (g \in V_1(\mathcal{D}))\,.$$

Then we have the following discrete analogue of the formula of integration by parts.

Lemma 2.2 (summation by parts). *For a scalar spring-block system $(\mathcal{D},\kappa)$, the equality:*

$$(u,v)_\kappa = \sum_{i=1}^{N} v_i \left(\sum_{j\in\Lambda_i} \kappa_{ij}(u_i - u_j)\right)$$

holds for all $u, v \in V(\mathcal{D})$.

Proof. From (2.3), we have

$$\begin{aligned}
(u,v)_\kappa &= \sum_{(i,j)\in\Lambda} \kappa_{ij}(u_j - u_i)(v_j - v_i)\\
&= \sum_{(i,j)\in\Lambda} \kappa_{ij}(u_j - u_i)v_j + \sum_{(i,j)\in\Lambda} \kappa_{ij}(u_i - u_j)v_i\\
&= \sum_{(j,i)\in\Lambda} \kappa_{ij}(u_i - u_j)v_i + \sum_{(i,j)\in\Lambda} \kappa_{ij}(u_i - u_j)v_i\\
&= \sum_{i=1}^{N} v_i \left(\sum_{j\in\Lambda_i} \kappa_{ij}(u_i - u_j)\right).
\end{aligned}$$

□

Using the summation by parts, we can derive a weak form of Problem 2.1.

Proposition 2.3. *Problem 2.1 is equivalent to the problem: Find $u \in V(\mathcal{D},g)$ such that*

$$(u,w)_\kappa = (f,w)_0 \quad ({}^\forall w \in V_0(\mathcal{D})). \tag{2.4}$$

Proof. For arbitrary $w \in V_0(\mathcal{D})$, we have the equality:

$$(f,w)_0 = \sum_{i\in J_0} F_i w_i. \tag{2.5}$$

From Lemma 2.2, we have

$$(u,w)_\kappa = \sum_{i\in J_0} \left(\sum_{j\in\Lambda_i} \kappa_{ij}(u_i - u_j)\right) w_i, \tag{2.6}$$

for any $u \in V(\mathcal{D})$. If $u \in V(\mathcal{D}, g)$ is a solution of Problem 2.1, the right hand sides of (2.5) and (2.6) are equal and (2.4) follows. Conversely, if $u \in V(\mathcal{D}, g)$ satisfies (2.4), the left hand sides of (2.5) and (2.6) are equal and (2.2) follows, since $w_i \in \mathbb{R}$ is arbitrary for $i \in J_0$. □

Concerning the solvability of Problem 2.1, we introduce some non-degenerate conditions of the spring constant κ. We define

$$c_0 = c_0(\mathcal{D}, \kappa, J) := \inf_{v \in V_0(\mathcal{D}),\, \|v\|_0 \neq 0} \frac{|v|_\kappa}{\|v\|_0} \geq 0.$$

Definition 2.4. Let $(\mathcal{D}, \kappa, J)$ be a scalar spring-block system with Dirichlet boundary.

1. $(\mathcal{D}, \kappa, J)$ is called *positively connected* if the following condition is satisfied:

$$v \in V_0(\mathcal{D}) \text{ and } \sum_{\kappa_{ij} > 0} |v_j - v_i| = 0, \text{ iff } v = 0 \in V(\mathcal{D}). \tag{2.7}$$

2. $(\mathcal{D}, \kappa, J)$ is called *regular* if $c_0(\mathcal{D}, \kappa, J) > 0$.

The condition (2.7) means that all the blocks D_i $(i \in J_0)$ is connected to a Dirichlet boundary block D_j $(j \in J_1)$ by a chain of springs of positive $\kappa_{ij} > 0$. We also remark that, if $(\mathcal{D}, \kappa, J)$ is regular, then the inequality

$$\|v\|_0 \leq c_0^{-1} |v|_\kappa \quad (v \in V_0(\mathcal{D})) \tag{2.8}$$

holds.

Proposition 2.5. *For a scalar spring-block system with Dirichlet boundary $(\mathcal{D}, \kappa, J)$, it is regular if and only if it is positively connected.*

Proof. We first remark that, since $V_0(\mathcal{D})$ is finite dimensional, it is not difficult to show existence of $\bar{v} \in V_0(\mathcal{D})$ which satisfies $\|\bar{v}\|_0 = 1$ and $|\bar{v}|_\kappa = c_0$.

We suppose that $(\mathcal{D}, \kappa, J)$ is positively connected. If it is not regular, there exists $\bar{v} \in V_0(\mathcal{D})$ such that $\|\bar{v}\|_0 = 1$ and $|\bar{v}|_\kappa = c_0 = 0$. But this contradicts the assumption that $(\mathcal{D}, \kappa, J)$ is positively connected. Hence $(\mathcal{D}, \kappa, J)$ is regular.

Next, we suppose that $(\mathcal{D}, \kappa, J)$ is regular. If $v \in V_0(\mathcal{D})$ satisfies the condition $\sum_{\kappa_{ij} > 0} |v_j - v_i| = 0$, then $|v|_\kappa = 0$ holds and $v = 0 \in V(\mathcal{D})$ follows from the inequality $\|v\|_0 \leq c_0^{-1} |v|_\kappa = 0$. Hence $(\mathcal{D}, \kappa, J)$ is positively connected. □

Lemma 2.6. *If u is a solution of Problem 2.1, then the following equality holds:*

$$E(v) - E(u) = \frac{1}{2} |v - u|_\kappa^2 \qquad (v \in V(\mathcal{D}, g)).$$

Proof. For $v \in V(\mathcal{D}, g)$, we set $w := v - u \in V_0(\mathcal{D})$. From Proposition 2.3, we obtain

$$\begin{aligned} E(v) - E(u) &= \frac{1}{2} |v|_\kappa^2 - \frac{1}{2} |u|_\kappa^2 - (f, v - u)_0 = \frac{1}{2} (v + u, v - u)_\kappa - (f, w)_0 \\ &= \frac{1}{2} (v + u, w)_\kappa - (u, w)_\kappa = \frac{1}{2} (v - u, w)_\kappa = \frac{1}{2} |v - u|_\kappa^2. \end{aligned}$$

□

Theorem 2.7. *Let $(\mathcal{D}, \kappa, J)$ be a regular scalar spring-block system with Dirichlet boundary. Then there exists a unique solution $u \in V(\mathcal{D})$ to Problem 2.1. Moreover, the solution u is a unique minimizer of $E(v)$ in $V(\mathcal{D}, g)$:*

$$u = \arg\min_{v \in V(\mathcal{D},g)} E(v), \tag{2.9}$$

and it satisfies the following estimates for all $v \in V(\mathcal{D}, g)$:

$$|u|_\kappa \leq |v|_\kappa + \frac{\|f\|_0}{c_0}, \tag{2.10}$$

$$\|u\|_0 \leq \|v\|_0 + \frac{2|v|_\kappa}{c_0} + \frac{\|f\|_0}{c_0^2}. \tag{2.11}$$

Proof. For $u \in V(\mathcal{D}, g)$, we set $\tilde{u} := u - g \in V_0(\mathcal{D})$. From Proposition 2.3, Problem 2.1 is equivalent to

$$(\tilde{u}, w)_\kappa = l(w) \quad ({}^\forall w \in V_0(\mathcal{D})), \tag{2.12}$$

where l is a linear functional on $V_0(\mathcal{D})$ defined by $l(w) := (f, w)_0 - (g, w)_\kappa$. Since $(\mathcal{D}, \kappa, J)$ is regular, $c_0 = c_0(\mathcal{D}, \kappa, J) > 0$ and the bilinear form $(\cdot, \cdot)_\kappa$ is coercive on $V_0(\mathcal{D})$, namely,

$$(w, w)_\kappa \geq c_0^2 \|w\|_0^2 \quad (w \in V_0(\mathcal{D})).$$

From the Lax-Milgram theorem, there uniquely exists $\tilde{u}$ which satisfies (2.12). Hence, the unique existence of the solution u of Problem 2.1 is obtained.

From Lemma 2.6, the solution u becomes a minimizer of the energy E among $V(\mathcal{D}, g)$. Conversely, if $u \in V(\mathcal{D}, g)$ is a minimizer of E among $V(\mathcal{D}, g)$, taking the first variation of the energy, for arbitrary $w \in V_0(\mathcal{D})$, we obtain

$$0 = \left.\frac{d}{d\varepsilon} E(u + \varepsilon w)\right|_{\varepsilon=0} = (u, w)_\kappa - (f, w)_0.$$

Hence, u is a solution of (2.4).

From Proposition 2.3, the solution u is decomposed as $u = u^1 + u^2$, where

$$u^1 \in V_0(\mathcal{D}) \quad \text{s.t.} \quad (u^1, v)_\kappa = (f, v)_0 \quad (v \in V_0(\mathcal{D})), \tag{2.13}$$

$$u^2 \in V(\mathcal{D}, g) \quad \text{s.t.} \quad (u^2, v)_\kappa = 0 \quad (v \in V_0(\mathcal{D})). \tag{2.14}$$

From (2.13), we have

$$|u^1|_\kappa^2 = (f, u^1)_0 \leq \|f\|_0 \|u^1\|_0 \leq c_0^{-1} \|f\|_0 |u^1|_\kappa.$$

Hence, we obtain

$$|u^1|_\kappa \leq c_0^{-1} \|f\|_0. \tag{2.15}$$

On the other hand, since u^2 is a unique minimizer of $E(v)$ among $v \in V(\mathcal{D}, g)$ with $f = 0$, we obtain

$$|u^2|_\kappa \leq |v|_\kappa \quad \text{for all} \quad v \in V(\mathcal{D}, g). \tag{2.16}$$

The inequality (2.10) follows from (2.15) and (2.16). The estimate (2.11) is also obtained as follows:

$$\begin{aligned}\|u\|_0 &\le \|v\|_0 + \|v-u\|_0 \le \|v\|_0 + c_0^{-1}|v-u|_\kappa \\ &\le \|v\|_0 + c_0^{-1}(|v|_\kappa + |u|_\kappa) \le \|v\|_0 + c_0^{-1}(2|v|_\kappa + c_0^{-1}\|f\|_0).\end{aligned}$$

□

2.3 tensor-valued spring constant model

In a similar way to the scalar spring constant model, we construct a tensor-valued spring constant model in this section.

For a block division $\mathcal{D}$ of Ω in $\mathbb{R}^n$, We consider a vector valued displacement $u = \sum_{i=1}^N u_i\chi_i \in V(\mathcal{D})^n$, where $u_i \in \mathbb{R}^n$ is a column vector and

$$V(\mathcal{D})^n := \left\{ v \in L^\infty(\Omega;\mathbb{R}^n);\ v = \sum_{i=1}^N v_i\chi_i,\ v_i \in \mathbb{R}^n \right\}.$$

For $(i,j) \in \Lambda$, We consider a virtual spring between the adjacent blocks D_i and D_j with tensor-valued spring constant $K_{ij} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, where $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ denotes a space of real symmetric matrices of size n. We suppose the condition:

$$K_{ij} = K_{ji} \ge O \qquad ((i,j) \in \Lambda),$$

where $K_{ij} \ge O$ means that K_{ij} is nonnegative definite. If $K_{ij} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ is positive definite, we denote it by $K_{ij} > O$. We also define

$$K := \sum_{(i,j)\in\Lambda} K_{ij}\chi_{ij} \in W(\mathcal{D})^{n\times n}.$$

Under the above conditions, we call $(\mathcal{D}, K)$ *a tensor-valued spring-block system*, and call $(\mathcal{D}, K, J)$ *a tensor-valued spring-block system with Dirichlet boundary.*

We consider the following problem.

Problem 2.8. *Let $(\mathcal{D}, K, J)$ be a tensor-valued spring-block system with Dirichlet boundary in $\mathbb{R}^n$. For a given body force $f = \sum f_i\chi_i \in V_0(\mathcal{D})^n$ with $F_i := |D_i|f_i \in \mathbb{R}^n$ and a given displacement $g = \sum g_i\chi_i \in V_1(\mathcal{D})^n$, find a displacement $u = \sum u_i\chi_i \in V(\mathcal{D})^n$ such that*

$$\begin{cases} \displaystyle\sum_{j\in\Lambda_i} K_{ij}(u_j - u_i) + F_i = 0 & (i \in J_0), \\ u_i = g_i & (i \in J_1). \end{cases} \tag{2.17}$$

We introduce the following symmetric bilinear form and seminorm:

$$\begin{aligned}(u,v)_K &:= \sum_{(i,j)\in\Lambda} \{K_{ij}(u_j - u_i)\}\cdot(v_j - v_i) \quad (u,\ v \in V(\mathcal{D})^n), \\ |v|_K &:= \sqrt{(v,v)_K} \quad (v \in V(\mathcal{D})).\end{aligned}$$

For Problem 2.8, we consider the following elastic energy of the springs with the outer force and an affine space for the Dirichlet boundary condition:

$$E(v) := \frac{1}{2}|v|_K^2 - (f, v)_0 \qquad (v \in V(\mathcal{D})^n),$$

$$V^n(\mathcal{D}, g) := \{v \in V(\mathcal{D})^n;\ v - g \in V_0(\mathcal{D})^n\} \qquad (g \in V_1(\mathcal{D})^n).$$

The summation by parts formula is valid even for the tensor-valued model.

Lemma 2.9 (summation by parts). *For a tensor-valued spring-block system $(\mathcal{D}, K)$, the equality:*

$$(u, v)_K = \sum_{i=1}^{N} v_i \cdot \left(\sum_{j \in \Lambda_i} K_{ij}(u_i - u_j) \right)$$

holds for all $u, v \in V(\mathcal{D})^n$.

Proposition 2.10. *Problem 2.8 is equivalent to the problem: Find $u \in V^n(\mathcal{D}, g)$ such that*

$$(u, w)_K = (f, w)_0 \quad ({}^\forall w \in V_0(\mathcal{D})^n).$$

Concerning the solvability of Problem 2.8, we introduce some non-degenerate conditions of the spring constant K. We define

$$c_0 = c_0(\mathcal{D}, K, J) := \inf_{v \in V_0(\mathcal{D})^n,\ \|v\|_0 \neq 0} \frac{|v|_K}{\|v\|_0} \geq 0.$$

Definition 2.11. Let $(\mathcal{D}, K, J)$ be a tensor-valued spring-block system with Dirichlet boundary.

1. $(\mathcal{D}, K, J)$ is called *positively connected* if the following condition is satisfied:

$$v \in V_0(\mathcal{D}) \text{ and } \sum_{K_{ij} > O} |v_j - v_i| = 0, \text{ iff } v = 0 \in V(\mathcal{D}). \tag{2.18}$$

2. $(\mathcal{D}, K, J)$ is called *regular* if $c_0(\mathcal{D}, K, J) > 0$.

The condition (2.18) means that all the blocks D_i $(i \in J_0)$ is connected to a Dirichlet boundary block D_j $(j \in J_1)$ by a chain of springs of positive definite $K_{ij} > O$. We also remark that, if $(\mathcal{D}, K, J)$ is regular, then the inequality

$$\|v\|_0 \leq c_0^{-1}|v|_K \quad (v \in V_0(\mathcal{D})^n) \tag{2.19}$$

holds.

Proposition 2.12. *For a tensor-valued spring-block system with Dirichlet boundary $(\mathcal{D}, K, J)$, it is regular if it is positively connected.*

Proof. We suppose that $(\mathcal{D}, K, J)$ is positively connected. If it is not regular, there exists $\bar{v} \in V_0(\mathcal{D})^n$ such that $\|\bar{v}\|_0 = 1$ and $|\bar{v}|_K = c_0 = 0$. But this contradicts the assumption that $(\mathcal{D}, K, J)$ is positively connected. Hence $(\mathcal{D}, K, J)$ is regular. □

In contrast with the scalar spring-block system, a regular tensor-valued spring-block system is not necessarily positively connected.

Lemma 2.13. *If u is a solution of Problem 2.8, then the following equality holds:*

$$E(v) - E(u) = \frac{1}{2}|v - u|_K^2 \qquad (v \in V^n(\mathcal{D}, g)).$$

Theorem 2.14. *Let $(\mathcal{D}, K, J)$ be a regular tensor-valued spring-block system with Dirichlet boundary. Then there exists a unique solution $u \in V(\mathcal{D})^n$ to Problem 2.8. Moreover, the solution u is a unique minimizer of $E(v)$ in $V^n(\mathcal{D}, g)$:*

$$u = \underset{v \in V^n(\mathcal{D},g)}{\arg\min}\ E(v), \tag{2.20}$$

and it satisfies the following estimates for all $v \in V^n(\mathcal{D}, g)$:

$$|u|_K \leq |v|_K + \frac{\|f\|_0}{c_0},$$
$$\|u\|_0 \leq \|v\|_0 + \frac{2|v|_K}{c_0} + \frac{\|f\|_0}{c_0^2}.$$

We omit proofs of Lemma 2.9, Proposition 2.10, Lemma 2.13 and Theorem 2.14, since they are shown in similar arguments to the scalar spring constant model.

3 Phase field model of fracture

3.1 Damage variable and phase field model

We construct a mathematical model of fracture on the scalar or tensor-valued spring constant model by introducing a damage variable. We represent the fracture or crack propagation by giving damage to the spring constant and cutting the spring according to the given damage.

For $(i, j) \in \Lambda$, the damage of the spring between the adjacent blocks D_i and D_j is assumed to be represented by $z_{ij}(t) \in [0, 1]$ at time t. We set $z_{ij} = 0$ if a spring is nondamaged, and set $z_{ij} = 1$ if it is completely broken. We also allow that z_{ij} takes an intermediate value in $(0, 1)$ if the spring is slightly damaged. We define

$$z(t) = \sum_{(i,j)\in\Lambda} z_{ij}(t)\chi_{ij} \in W(\mathcal{D}),$$

and call $z(t)$ *a damage variable* or *a phase field of damage*. We define sets of damage variables:

$$\mathcal{Z} := \left\{ \zeta = \sum_{(i,j)\in\Lambda} \zeta_{ij}\chi_{ij} \in W(\mathcal{D}),\ \zeta_{ij} \in [0,1] \right\},$$

$$\mathcal{Z}_0 := \left\{ \zeta \in \mathcal{Z};\ v \in V_0(\mathcal{D}) \text{ and } \sum_{\zeta_{ij}\neq 1} |v_j - v_i| = 0, \text{ iff } v = 0 \in V(\mathcal{D}) \right\},$$

$$\mathcal{Z}_1 := \{\zeta \in \mathcal{Z};\ \zeta_{ij} \in [0,1)\}.$$

If $z(t) \in \mathcal{Z}_0$, it means that each block D_i of $i \in J_0$ is connected with a block D_j of Dirichlet boundary ($j \in J_1$) by some springs which are not completely bloken. We also remark that $\mathcal{Z}_1 \subset \mathcal{Z}_0$.

For a given scalar spring constant $\kappa = \sum_{(i,j)\in\Lambda} \kappa_{ij}\chi_{ij}$, the damaged spring constant $\tilde{\kappa}(t) = \sum_{(i,j)\in\Lambda} \tilde{\kappa}_{ij}(t)\chi_{ij}$ is defined by

$$\tilde{\kappa}_{ij}(t) := \eta(z_{ij}(t))\kappa_{ij} \qquad ((i,j) \in \Lambda),$$

where η is a given function which satisfies the conditions:

$$\eta \in C^0([0,\infty)) \cap C^2([0,1)), \quad \eta(0) = 1, \quad \eta'(s) < 0 \ \ (0 \le s < 1), \quad \eta(s) = 0 \ \ (s \ge 1).$$

In case of a tensor-valued spring-block system, we define the damaged spring constant $\tilde{K}(t) = \sum_{(i,j)\in\Lambda} \tilde{K}_{ij}(t)\chi_{ij}$ is defined by

$$\tilde{K}_{ij}(t) := \eta(z_{ij}(t))K_{ij} \qquad ((i,j) \in \Lambda).$$

For the damaged spring-block systems, we have the following propositions.

Proposition 3.1. *Let $(\mathcal{D}, \kappa, J)$ be a scalar spring-block system with Dirichlet boundary. For a damage variable $z \in \mathcal{Z}$, we define a damaged spring constant $\tilde{\kappa} = \sum_{(i,j)\in\Lambda} \tilde{\kappa}_{ij}\chi_{ij}$ by $\tilde{\kappa}_{ij} := \eta(z_{ij})\kappa_{ij}$.*

1. *We suppose that $(\mathcal{D}, \kappa, J)$ is regular. Then $(\mathcal{D}, \tilde{\kappa}, J)$ is regular if $z \in \mathcal{Z}_1$.*
2. *We suppose that $\kappa_{ij} > 0$ for all $(i,j) \in \Lambda$. Then $(\mathcal{D}, \tilde{\kappa}, J)$ is regular if and only if $z \in \mathcal{Z}_0$.*

Proof. For the first statement, we set

$$z^* := \max_{(i,j)\in\Lambda} z_{ij} < 1.$$

Then we have $\eta(z_{ij}) \ge \eta(z^*) > 0$ for all $(i,j) \in \Lambda$. For $v \in V_0(\mathcal{D})$, since

$$|v|_{\tilde{\kappa}}^2 = \sum_{(i,j)\in\Lambda} \eta(z_{ij})\kappa_{ij}(v_j - v_i)^2 \ge \eta(z^*)|v|_K^2 \ge \eta(z^*)c_0(\mathcal{D}, \kappa, J)^2\|v\|_0^2,$$

$(\mathcal{D}, \tilde{\kappa}, J)$ is regular. The second statement is also shown by virtue of Proposition 2.5, since $(\mathcal{D}, \tilde{\kappa}, J)$ is positively connected if and only if $z \in \mathcal{Z}_0$. $\square$

Proposition 3.2. *Let $(\mathcal{D}, K, J)$ be a tensor-valued spring-block system with Dirichlet boundary. For a damage variable $z \in \mathcal{Z}$, we define a damaged spring constant $\tilde{K} = \sum_{(i,j)\in\Lambda} \tilde{K}_{ij}\chi_{ij}$ by $\tilde{K}_{ij} := \eta(z_{ij})K_{ij}$.*

1. *We suppose $(\mathcal{D}, K, J)$ is regular. Then $(\mathcal{D}, \tilde{K}, J)$ is regular if $z \in \mathcal{Z}_1$.*
2. *We suppose that $K_{ij} > 0$ for all $(i,j) \in \Lambda$. Then $(\mathcal{D}, \tilde{K}, J)$ is regular if $z \in \mathcal{Z}_0$.*

We can prove this proposition in the same manner of the proof of Proposition 3.1.

We define

$$\varphi(s) := \begin{cases} -\frac{1}{2}\eta'(s) & (0 \le s < 1) \\ 0 & (s \ge 1) \end{cases}.$$

A typical choice of η and φ is

$$\eta(s) = ((1-s)_+)^2, \qquad \varphi(s) = (1-s)_+,$$

where $(a)_+ = \max(0, a)$. This η belongs to $C^1([0,\infty)) \cap W^{2,\infty}(0,\infty)$. Another example is

$$\eta(s) = (1-s)_+, \qquad \varphi(s) = \begin{cases} \frac{1}{2} & (0 \le s < 1) \\ 0 & (s \ge 1) \end{cases}.$$

We suppose that the crack propagation speed is slow and the quasi-stationary state for the displacement field $u(t) \in V(\mathcal{D})$ is approximately valid during fracture progress. For each time t, We consider the force balance equations with the modified spring constant. For the damage variable $z(t)$, we consider the following model:

$$\alpha \frac{dz_{ij}}{dt} = \varphi(z_{ij})\big(Q_{ij} - \gamma_{ij}\big)_+ \qquad ((i,j) \in \Lambda), \tag{3.1}$$

where

$$Q_{ij}(t) := \kappa_{ij}(u_j(t) - u_i(t))^2, \quad \text{or} \quad Q_{ij}(t) := \{K_{ij}(u_j(t) - u_i(t))\} \cdot (u_j(t) - u_i(t)) \tag{3.2}$$

represents the magnitude of the strain energy between D_i and D_j in case of the scalar or tensor-valued case, respectively. The given constant $\gamma_{ij} > 0$ corresponds to a strength of the spring. The parameter $\alpha > 0$ stands for a time constant of time relaxation effect. In our model (3.1), the damage variable z_{ij} tends to 1 if the strain energy Q_{ij} exceeds the given threshold γ_{ij}, however z_{ij} does not change if $Q_{ij} \le \gamma_{ij}$.

In a usual elastic material, a crack once appeared in the material does not heal by itself. We also suppose this non-repair condition of the crack in our model. By virtue of the form $\alpha\frac{dz_{ij}}{dt} = (\cdot)_+$, the damage variable is non-decreasing in t, which represents the non-repair condition.

We consider the following conditions for the body force, the boundary displacement and the initial damage. For $l \in \{0,1,2\}$, we suppose

$$f = \sum f_i\chi_i \in C^l([0,\infty), V_0(\mathcal{D})), \qquad g = \sum g_i\chi_i \in C^l([0,\infty), V_1(\mathcal{D})), \qquad z^0 \in \mathcal{Z}, \tag{3.3}$$

where, in the case of tensor-valued spring-block system, we suppose $f_i(t) \in V_0(\mathcal{D})^n$ and $g_i(t) \in V_1(\mathcal{D})^n$. Hence, we consider the following problems.

Problem 3.3. *Let $(\mathcal{D}, \kappa, J)$ be a scalar spring-block system with Dirichlet boundary in $\mathbb{R}^n$. For given f, g and z^0 with the condition* (3.3)*, find a displacement $u(t) = \sum u_i(t)\chi_i \in V(\mathcal{D})$ for a.e. $t \in [0,T)$ and a damage variable $z \in C^0([0,T), \mathcal{Z})$ with $\frac{dz}{dt} \in L^1(0,T; W(\mathcal{D}))$ for some $T \in (0,\infty]$ such that*

$$\begin{cases} \sum_{j\in\Lambda_i} \tilde{\kappa}_{ij}(t)(u_j(t) - u_i(t)) + F_i(t) = 0 & (i \in J_0,\ t \in [0,T)), \\ u_i(t) = g_i(t) & (i \in J_1,\ t \in [0,T)), \\ \alpha\dfrac{dz_{ij}}{dt}(t) = \varphi(z_{ij}(t))\big(Q_{ij}(t) - \gamma_{ij}\big)_+ & ((i,j) \in \Lambda,\ \text{a.e. } t \in [0,T)), \\ z_{ij}(0) = z_{ij}^0 & ((i,j) \in \Lambda), \end{cases} \tag{3.4}$$

where $F_i(t) := |D_i| f_i(t)$ for $i = 1, \cdots, N$.

Problem 3.4. *Let $(\mathcal{D}, K, J)$ be a tensor-valued spring-block system with Dirichlet boundary in $\mathbb{R}^n$. For given f, g and z^0 with the condition* (3.3)*, find a displacement $u(t) = \sum u_i(t)\chi_i \in V(\mathcal{D})^n$ for a.e. $t \in [0,T)$ and a damage variable $z \in C^0([0,T), \mathcal{Z})$ with $\frac{dz}{dt} \in L^1(0,T; W(\mathcal{D}))$ for some $T \in (0,\infty]$ such that*

$$\begin{cases} \sum_{j\in\Lambda_i} \tilde{K}_{ij}(t)(u_j(t) - u_i(t)) + F_i(t) = 0 & (i \in J_0,\ t \in [0,T)), \\ u_i(t) = g_i(t) & (i \in J_1,\ t \in [0,T)), \\ \alpha\dfrac{dz_{ij}}{dt}(t) = \varphi(z_{ij}(t))\big(Q_{ij}(t) - \gamma_{ij}\big)_+ & ((i,j) \in \Lambda,\ \text{a.e. } t \in [0,T)), \\ z_{ij}(0) = z_{ij}^0 & ((i,j) \in \Lambda), \end{cases}$$

where $F_i(t) := |D_i| f_i(t)$ for $i = 1, \cdots, N$.

A numerical example of a simulation of Problem 3.4 is shown in Figure 3, where we give a crack opening load to an initially cracked plate. If $z_{ij}(t) \geq 1 - \varepsilon$ for small $\varepsilon > 0$, the spring between the blocks D_i and D_j is almost broken and we consider $D_{ij} = \overline{D_i} \cap \overline{D_j}$ is a part of the crack and bold it in the figures. A close view of a crack tip is shown in Figure 2. In Figure 3, we can observe that a straight crack propagates in time.

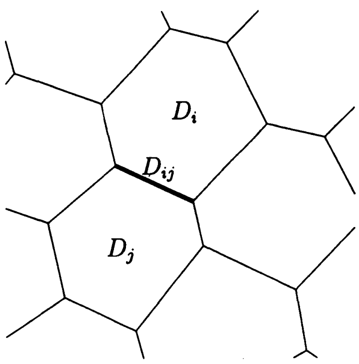

Figure 1: An adjacent blocks D_i and D_{ij}

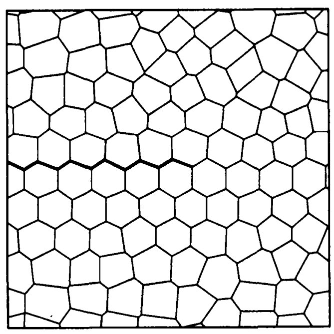

Figure 2: Close view of a crack tip

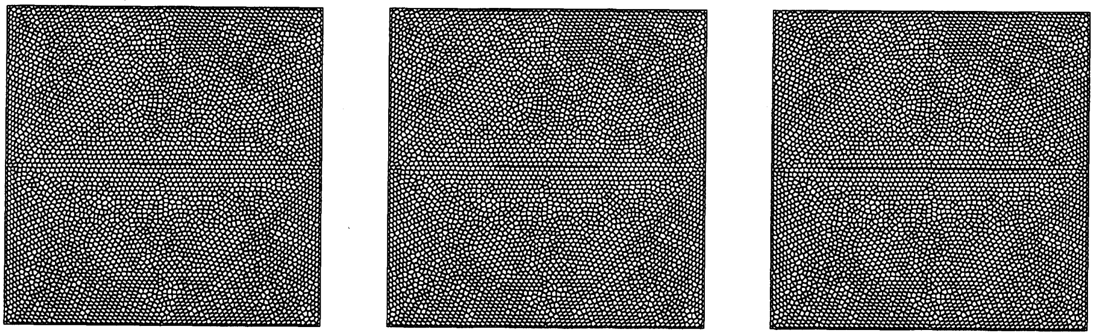

Figure 3: Example of crack propagation on a tensor-valued spring constant model: Initial configuration (left), Final configuration (right).

3.2 Solvability and regularity

Since the initial value problem (3.4) may have a singularity, we consider $W^{1,1}$-solution in Problem 3.3 instead of the standard C^1-solution. Actually, (3.4) is considered as a system of ODEs of $\{z_{ij}(t)\}_{(i,j)\in\Lambda}$, and a singularity may exist at $z_{ij} = 1$ or if the coefficient matrix of the linear system of the displacement field $\{u_i\}_{i\in J_0}$ is singular. We state our mathematical results in the following three theorems in case of the scalar spring-block system. We, however, remark that these theorems are valid even for Problem 3.4.

Theorem 3.5. *We suppose the condition* (3.3) *with* $l = 0$. *If* $(\mathcal{D}, \tilde{\kappa}(0), J)$ *is regular, then there exist* $T_0 \in (0,\infty)$ *and a solution* $(u(t), z(t))$ *for* $0 \le t \le T_0$ *to Problem 3.3, and the solution is unique in the time interval* $[0,T_0]$. *It also satisfies that* $u \in C^0([0,T_0], V(\mathcal{D}))$ *and* $z \in C^1([0,T_0],\mathcal{Z})$, *and that* $(\mathcal{D},\tilde{\kappa}(t),J)$ *is regular for* $t \in [0,T_0]$. *Furthermore, if* $l = 1$, *then* $u \in C^1([0,T_0],V(\mathcal{D}))$ *and* $z \in W^{2,\infty}(0,T_0;\mathcal{Z})$ *hold. If* $l = 2$, *then* $u \in W^{2,\infty}(0,T_0;V(\mathcal{D}))$ *holds.*

Proof. Since $(\mathcal{D},\tilde{\kappa}(0),J)$ is regular, from Theorem 2.7, $u(0)$ is uniquely determined from the linear system of the first two equations of (3.4). From the Cramer's formula, $u(0)$ is represented in the form:

$$u_k(0) = \frac{p_k(\tilde{\kappa}(0),\ f(0),\ g(0))}{p_0(\tilde{\kappa}(0))} \quad (k = 1,\dots,N), \tag{3.5}$$

where p_0 is the determinant given as a polynomial of $\tilde{\kappa}_{ij}(0)$ and p_k is also a polynomial of $\tilde{\kappa}_{ij}(0)$, $f_i(0)$ and $g_i(0)$.

We define

$$\Lambda^* := \{(i,j)\in\Lambda;\ 0 \le z^0_{ij} < 1\},$$
$$\mathcal{Z}^* := \{\zeta\in\mathcal{Z};\ \zeta_{ij} = 1 \quad ((i,j)\in\Lambda\setminus\Lambda^*)\}.$$

It is sufficient to consider $z(t)\in\mathcal{Z}^*$ since $z_{ij}(t)$ is non-decreasing.

For $\zeta \in \mathcal{Z}^*$ and $t \geq 0$, we define

$$\bar{\kappa}(\zeta) := \sum_{(i,j)\in\Lambda} \eta(\zeta_{ij})\kappa_{ij}\chi_{ij} \in W(\mathcal{D}),$$
$$\bar{u}_k(\zeta,t) := \frac{p_k(\bar{\kappa}(\zeta),\ f(t),\ g(t))}{p_0(\bar{\kappa}(\zeta))} \quad (k=1,\ldots,N),$$
$$\bar{Q}_{ij}(\zeta,t) := \kappa_{ij}(\bar{u}_j(\zeta,t)-\bar{u}_i(\zeta,t))^2 \quad ((i,j)\in\Lambda),$$

and we consider the following system of ODEs of $z_{ij}(t)$ for $(i,j) \in \Lambda^*$.

$$\begin{cases} \alpha\dfrac{dz_{ij}}{dt}(t) = \varphi(z_{ij}(t))\big(\bar{Q}_{ij}(z(t),t)-\gamma_{ij}\big)_+ & ((i,j)\in\Lambda^*,\ t\geq 0), \\ z_{ij}(0) = z_{ij}^0 & ((i,j)\in\Lambda^*), \\ z_{ij}(t) = 1 & ((i,j)\in\Lambda\setminus\Lambda^*). \end{cases} \tag{3.6}$$

From the standard theory of ODE, since our system (3.6) satisfies the Lipschitz condition, it follows that there exists a unique local solution $z \in C^1([0,T_0],\mathcal{Z}^*)$, with some $T_0 > 0$. Without loss of generality, we can assume that $z_{ij}(t) \in [0,1)$ for all $(i,j) \in \Lambda^*$ and $t \in [0,T_0]$. We set $u(t) := \bar{u}(z(t),t)$. Then $(u(t),z(t))$ is becomes a solution of Problem 3.3. It is clear that this is a unique solution of Problem 3.3 in the time interval $[0,T_0]$.

Moreover, since $p_0(\tilde{\kappa}(t)) = p_0(\bar{\kappa}(z(t))) \neq 0$, it follows that $(\mathcal{D},\tilde{\kappa}(t),J)$ is regular for $t \in [0,T_0]$. Under the above conditions, it also follows that $\tilde{\kappa}_{ij} \in C^1([0,T_0])$ and $\varphi \circ z_{ij} \in C^1([0,T_0])$. In particular, from (3.5), $u \in C^0([0,T_0],V(\mathcal{D}))$ follows.

If (3.3) holds for $l = 1$, from (3.5), we obtain that $u \in C^1([0,T_0],V(\mathcal{D}))$ and that $\frac{dz_{ij}}{dt} \in W^{1,\infty}(0,T_0;W(\mathcal{D}))$. We also have $z \in W^{2,\infty}(0,T_0;W(\mathcal{D}))$ and $\tilde{\kappa}_{ij} \in W^{2,\infty}(0,T_0)$.

If (3.3) holds for $l = 2$, $u \in W^{2,\infty}(0,T_0;V(\mathcal{D}))$ holds from (3.5). □

By replacing $\kappa_{ij} = 1$ and $z_{ij}^0 = 1$ in case of $\kappa_{ij} = 0$, we can assume that all spring constant κ_{ij} is positive without loss of generality. For a damage variable $z(t)$ $(0 \leq t < T \leq \infty)$, we define

$$J(i,t) := \left\{ k \in \{1,\cdots,N\};\ v_k = 0,\ \text{if } v \in V(\mathcal{D}) \text{ and } \sum_{z_{ij}(t)<1} |v_j - v_i| = 0 \right\},$$

for $i = 1,\cdots,N$, where $k \in J(i,t)$ means that the block D_k is connected by a chain of positive spring constants with the block D_i. We call $J(i,t)$ an index set of connected blocks to D_i.

Let $I(t)$ be the number of completely broken springs, namely,

$$I(t) := \#\{(i,j)\in\Lambda;\ z_{ij}(t) = 1\}.$$

There exists $0 = t_0 < t_1 < \cdots < t_q = T$ such that

$$I(t_0) < I(t_1) < \cdots < I(t_{q-1}), \quad I(t) = I(t_{m-1}) \quad (t \in [t_{m-1},t_m),\ m = 1,\cdots,q).$$

Then we also have

$$J(i,t) = J(i,t_{m-1}) \quad (i = 1,\cdots,N,\ t \in [t_{m-1},t_m),\ m = 1,\cdots,q). \tag{3.7}$$

For fixed $m = 1, \cdots, q$, in each time interval $[t_{m-1}, t_m)$, $\mathcal{D}$ is divided into subblock system $\mathcal{D}^1, \cdots, \mathcal{D}^p (\neq \emptyset)$, where $\mathcal{D}^k$ depends on m and

$$\mathcal{D} = \bigcup_{k=1}^{p} \mathcal{D}^k, \quad i_k := \min\{j;\ D_j \in \mathcal{D}^k\}, \quad \mathcal{D}^k = \{D_i\}_{i \in J(i_k, t_{m-1})}.$$

Theorem 3.6. *Under the condition* (3.3) *with* $l = 0$, *we suppose that there exists a solution* $(u(t), z(t))$ $(0 \leq t < T \leq \infty)$ *of Problem 3.3, and define* $0 = t_0 < t_1 < \cdots < t_q = T$ *with the condition* (3.7). *Then the solution satisfies the following properties.*

1. *The damage variable* $z(t)$ *is unique in the interval* $[0, T)$, *and* $z \in C^1([t_{m-1}, t_m), W(\mathcal{D}))$ *for* $m = 1, \cdots, q$.
2. *There exists* $\tilde{u}(t)$ *with* $\tilde{u} \in C^0([t_{m-1}, t_m), V(\mathcal{D}))$ *for* $m = 1, \cdots, q$ *such that* $(\tilde{u}(t), z(t))$ *is a solution of Problem 3.3.*
3. *Suppose that* $z_{ij}(t) \in [0, 1)$ *for* $0 \leq t < t_m$. *Then the quantity* $Q_{ij}(t)$ *for* $t \in [0, t_m)$ *is uniquely determined and* $Q_{ij} \in C^0([0, t_m))$.

Proof. For fixed $m = 1, \cdots, q$, in each interval $[t_m - 1, t_m)$, we consider reduced problems in each connected spring block system $\mathcal{D}^k$ $(k = 1, \cdots, p)$. We define $\tilde{J}_1^k := J(i_k, t_{m-1}) \cap J_1$, and set

$$J^k = (J_0^k, J_1^k), \quad J_0^k := J(i_k, t_{m-1}) \setminus J_1^k, \quad J_1^k := \begin{cases} \tilde{J}_1^k & \text{if } \tilde{J}_1^k \neq \emptyset \\ \{i_k\} & \text{if } \tilde{J}_1^k = \emptyset \end{cases}$$

Then $(\mathcal{D}^k, \tilde{\kappa}(t), J^k)$ becomes regular spring-block system with Dirichlet boundary for $t \in [t_{m-1}, t_m)$. Then we can solve (3.4) in each $\mathcal{D}^k$ with the initial condition $z(t_{m-1})$ for z and Dirichlet boundary condition:

$$g_i^k(t) = \begin{cases} g_i(t) & \text{if } \tilde{J}_1^k \neq \emptyset \\ 0 & \text{if } \tilde{J}_1^k = \emptyset \end{cases}, \quad (i \in J_1^k).$$

We define

$$\tilde{u}_i(t) := \begin{cases} u_i(t) & \text{if } \tilde{J}_1^k \neq \emptyset \\ u_i(t) - u_{i_k}(t) & \text{if } \tilde{J}_1^k = \emptyset \end{cases}, \quad (D_i \in \mathcal{D}^k,\ t \in [t_{m-1}, t_m)).$$

Then it is easy to show that $(\tilde{u}(t), z(t))$ is a solution of each regular sub-spring-block system. From Theorem 3.5, the solution is unique and it satisfies $\tilde{u} \in C^0([t_{m-1}, t_m), V(\mathcal{D}))$ and $z \in C^1([t_{m-1}, t_m), W(\mathcal{D}))$.

We also remark that $z(t)$ is globally unique, since $z(t_m) = \lim_{t \to t_m - 0} z(t)$ always exists due to the monotonicity of $z_{ij}(t)$ and we can extend $z(t)$ uniquely in the next interval $[t_m, t_{m+1})$.

If $(u(t), z(t))$ and $(\tilde{u}(t), \tilde{z}(t))$ are both solutions of Problem 3.3, from the uniqueness of $z(t)$, we have $\tilde{z}(t) = z(t)$. Then, for fixed t, $u(t) - \tilde{u}(t)$ becomes a solution of force balance equations with $f \equiv 0$ and $g \equiv 0$. Hence, we have $u_j(t) - \tilde{u}_j(t) = u_i(t) - \tilde{u}_i(t)$ if $j \in \Lambda(i, t)$. It holds that $u_j(t) - u_i(t) = \tilde{u}_j(t) - \tilde{u}_i(t)$ if $z_{ij}(t) \in [0, 1)$, and the third claim follows. □

Theorem 3.7. *We suppose that $f \equiv 0$, $g \in C^0([0,\infty), V_1(\mathcal{D}))$ and $z^0 \in \mathcal{Z}$. Then there exists a global solution $(u(t), z(t))$ to Problem 3.3 in $0 \le t < \infty$.*

Proof. We suppose that $\kappa_{ij} > 0$ for all $(i,j) \in \Lambda$ without loss of generality. Similarly to the proof of Theorem 3.6, we can construct a solution $(u(t), z(t))$ and $0 = t_0 < t_1 < \cdots < t_q = \infty$ by solving each reduced problem in $\mathcal{D}^k$ with the initial condition at $t = t_{m-1}$. We remark that, even if $\tilde{J}_1^k = \emptyset$, $u_i(t) = 0$ for $D_i \in \mathcal{D}^k$ satisfies (3.4) since $f \equiv 0$, and $z_{ij}(t) = z_{ij}(t_{m-1})$ for $t \ge t_{m-1}$ if $J(i, t_{m-1}) \cap J(j,t) \cap J_1 = \emptyset$. $\square$

4 Conclusion

In this paper, we proposed a mathematical model of fracture of an elastic material. The deformation of the elastic body is approximated by a spring-block system and the crack is represented by a damage variable defined on each springs. We remark that this research is based on the idea of [6], and that a dynamic problem in similar setting is studied in [1].

Due to the page limitation, we could not describe some further results such as an energy decay property, a uniform estimate of the energy, an estimate of crack length and more numerical examples. We also could not discuss about the consistency of the scalar or tensor-valued spring constant model with the linear elasticity problem. They will be discussed in our forthcoming papers.

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